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ON THE COMPLETE INDEPENDENCE OF HURWITZ'S POSTULATES FOR ABELIAN GROUPS AND FIELDS.*

BY B. A. BERNSTEIN.

In these *Annals*, in 1913, Hurwitz presented sets of postulates for abelian groups and fields—three for abelian groups (finite, denumerably infinite, and non-denumerably infinite) and three for corresponding fields.† The chief characteristics of each of these sets are the simplicity of the statements, the small number of postulates used, and the elegance of the systems establishing (ordinary) independence.‡ The object of this paper is to consider for these admirable sets of postulates the question of complete independence,§ which question Professor Hurwitz left untouched.

Hurwitz's postulates. Hurwitz's postulates are found among the following eight conditions on a class K and two binary operations $\oplus$, $\odot$.

(A_1) If a , b , c , $a \oplus b$, $c \oplus b$, and $a \oplus (c \oplus b)$ belong to K , then $(a \oplus b) \oplus c = a \oplus (c \oplus b)$.

(A_2) If a and b belong to K , then there is an element x of K such that $a \oplus x = b$.

(M_1) If a , b , c , $a \odot b$, $c \odot b$, and $a \odot (c \odot b)$ belong to K , then $(a \odot b) \odot c = a \odot (c \odot b)$.

(M_2) If a and b belong to K , and $a \oplus a \neq a$, there is an element x of K such that $a \odot x = b$.

(D) If a , b , c , $a \odot b$, $a \odot c$, $b \oplus c$, $(a \odot b) \oplus (a \odot c)$ belong to K , then $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

(N_n) K contains n (> 1) elements.

(N') K is countably infinite.

* Read before the San Francisco Section of the American Mathematical Society, October 22, 1921.

† W. A. Hurwitz, "Postulate-sets for abelian groups and fields," these *Annals* (2), vol. 15 (1913), p. 93. Compare his "Note on the definition of an abelian group," the *Annals* (2), vol. 8 (1907), p. 94.

‡ The postulates are based on sets of postulates for abelian groups and fields given by Huntington. See E. V. Huntington, "Definitions of a field by sets of independent postulates," *Trans. Amer. Math. Soc.*, vol. 4 (1903), p. 31, and "Note on the definitions of abstract groups and fields by sets of independent postulates," *Trans. Amer. Math. Soc.*, vol. 6 (1905), p. 181. While retaining the elegance of Huntington's postulates, Hurwitz reduces their number by one for abelian groups and by two for fields.

§ Professor E. H. Moore first proposed the problem of complete independence of a set of postulates. See his "Introduction to a form of general analysis," *New Haven Mathematical Colloquium*, Yale University Press, p. 82. On the significance of the question of complete independence of postulates see also E. V. Huntington, *Bull. Amer. Math. Soc.*, vol. 23 (1917), p. 278, and J. S. Taylor, *Bull. Amer. Math. Soc.*, vol. 26 (1920), p. 449, footnote.

(N'') K has the cardinal number of the continuum.

Let G_n , G' , G'' , F_n , F' , F'' denote the sets taken from the above "matrix" as follows:

$$\begin{aligned} G_n &: (A_1), (A_2), (N_n), \\ G' &: (A_1), (A_2), (N'), \\ G'' &: (A_1), (A_2), (N''), \\ F_n &: (A_1), (A_2), (M_1), (M_2), (D), (N_n), \\ F' &: (A_1), (A_2), (M_1), (M_2), (D), (N'), \\ F'' &: (A_1), (A_2), (M_1), (M_2), (D), (N''). \end{aligned}$$

Hurwitz proves that G_n , G' , G'' form sets of independent postulates for abelian groups having respectively n elements, a countable infinity of elements, and elements whose cardinal number is that of the continuum; and he proves that F_n , F' , F'' form sets of independent postulates for corresponding fields.

Complete independence. The question of complete independence of the postulate-sets is answered by the following

THEOREM. *Postulate-sets F' , F'' , G' , G'' , G_n ($n > 1$) are each completely independent; postulate-set F_n is completely independent when, and only when, n exceeds 2 and is a power of a prime.*

To prove the complete independence of F' we take for systems having the characters $(\pm\pm\pm\pm\pm)$ systems 1-32 in Table A below. By taking for K in this table the class of *reals*, instead of the class of rationals, we obtain systems, 1'-32', having the characters $(\pm\pm\pm\pm\pm-)$.

That set F'' is completely independent is seen from the fact that, with respect to F'' , systems 1-32 have the characters $(\pm\pm\pm\pm\pm-)$, while systems 1'-32' have the characters $(\pm\pm\pm\pm\pm+)$.

Since G' is included in F' and G'' in F'' , postulate-sets G' , G'' are each completely independent.

Proof-systems showing the complete independence of G_n are systems 4, 5, 6, 16,† together with the systems obtained from 4, 5, 6, 16* by replacing (1) the class of rationals by the class of n integers 0, 1, $\dots$, $n-1$ ($n > 1$) and (2) the operation $a + b$ (in 4) by the operation $a + b \bmod n$.

In order to see that F_n is completely independent for every integer $n > 2$ and a power of a prime, we observe (1) that with respect to F_n systems 1-32 of Table A have the characters $(\pm\pm\pm\pm\pm-)$; (2) that the Galois field of order $n = q^k$, q prime and $n > 2$, gives the character $(++++++)$; and (3) that systems 2-32 will have the remaining 31 of the 32 characters $(\pm\pm\pm\pm\pm+)$ if in these systems we replace (1) the

* When n is a power of a prime.

† As far as K , $\oplus$ are concerned.

class of rationals by the class of n integers $0, 1, \dots, n-1$ ($n > 2$ and a power of a prime) and (2) the operation $a + b$ by $a + b \bmod n$.

TABLE A.
SYSTEMS HAVING THE CHARACTERS $(\pm \pm \pm \pm \pm \pm)$ FOR F' .

No.	Character.	K.	$a \oplus b$.	$a \odot b$.
1	(++++++)	Rationals	$a + b$	ab
2	(+++++)	"	$a + b$	$a + b$
3	(++++-)	"	$a + b$	0
4	(+++--)	"	$a + b$	b
5	(++---)	"	a	$a + b$
6	(+-+--)	"	b	$a + b$
7	(+--+)	"	$a + b$	1
8	(+---)	"	$a + b$	$b + 1$
9	(-++--)	"	0	$a + b$
10	(-++-)	"	b	0
			except: $2 \oplus 0 = 1$ $2 \oplus 1 = 0$	except: $1 \odot 1 = 1$
11	(++---)	"	$a + b$	b
				except: $1 \odot a = 0$
12	(+-+--)	"	0	0
13	(-++--)	"	$b + 1$	y^*
14	(+---)	"	0	b
15	(-++-)	"	b	b
16	(-+-+)	"	y^*	0
			except: $a \oplus a = a$ $0 \oplus 1 = 1$ $1 \oplus 0 = 0$	
17	(++---)	"	$a + b$	$a + 1$
18	(+-+--)	"	0	1
19	(-++--)	"	$b + 1$	1
20	(+---)	"	0	$b + 1$
21	(-++-)	"	b	0
			except: $2 \oplus 0 = 1$ $2 \oplus 1 = 0$	except: $2 \odot 0 = 1$
22	(---++)	"	0	$a + b$
			except: $1 \oplus 0 = 1$	
23	(+---)	"	0	0
				except: $0 \odot 1 = 1$
24	(-+---)	"	$b + 1$	b
				except: $0 \odot a = y^*$
25	(--+-+)	"	1	1
			except: $1 \oplus 0 = 0$	
26	(----++)	"	$a + 1$	b
27	(+-----)	"	0	$a + 1$
28	(-+-----)	"	$b + 1$	1
				except: $0 \odot 1 = 0$
29	(--+-+)	"	0	0
			except: $0 \oplus 0 = 1$	
30	(----++)	"	b	b
			except: $1 \oplus 1 = 0$	except: $1 \odot 1 = 0$
31	(----+)	"	0	$b + 1$
			except: $0 \oplus 0 = 1$	
32	(-----)	"	$a + 1$	$a + 1$

Finally, F_n is *not* completely independent when n is other than a power of a prime, or when $n = 2$, because (1) there exists no field for n

* Not an element of K .

other than a power of a prime, and (2) there exists no system of character $(-+-+ -+)$ when $n = 2$.* This completes the proof of our theorem.

If we only wish to prove the complete independence of sets F' , F'' , G' , G'' , systems 1°–32° of Table B below will be found more simple than systems 1–32 above.

TABLE B.
SYSTEMS HAVING THE CHARACTERS $(\pm \pm \pm \pm \pm +)$ FOR F' .

No.	Character.	K.	$a \oplus b$.	$a \odot b$.
1°	(++++++)	Rational†	$a + b$	ab
2°	(+++++)	"	$a + b$	$a + b$
3°	(++++-)	"	$a + b$	0
4°	(+++--)	"	$a + b$	b/a
5°	(++---)	"	0	ab
6°	(+----)	"	$a - b$	ab
7°	(-+---)	"	$a + b$	1
8°	(--+-)	"	$a + b$	$a - b$
9°	(---+)	"	0	$a + b$
10°	(----)	"	$a - b$	$a + b$
11°	(-+---)	"	$a + b$	$b/(a - 1)$
12°	(++---)	"	0	0
13°	(+----)	"	$a - b$	0
14°	(-+---)	"	0	b
15°	(--+-)	"	b	b
16°	(---+)	"	$a/2$	ab
17°	(----)	"	$a + b$	a/b
18°	(-+---)	"	0	1
19°	(++---)	"	$a - b$	1
20°	(+----)	"	0	$a - b$
21°	(-+---)	"	$a - b$	$a - b$
22°	(--+-)	"	a/b	$a + b$
23°	(---+)	"	0	$(a - 1)b$
24°	(----)	"	$a - b$	$b/(a - 1)$
25°	(-+---)	"	a/b	1
26°	(++---)	"	$a/2$	b/a
27°	(+----)	"	0	$a + 1$
28°	(-+---)	"	$a - b$	a/b
29°	(--+-)	"	a/b	ab
30°	(---+)	"	$a/2$	$(a - 1)b$
31°	(----)	"	a/b	$(a - 1)b$
32°	(-+---)	"	a/b	a/b

UNIVERSITY OF CALIFORNIA,
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* If 0, 1 be the two elements of K , the only choice we have for $a \oplus b$ so that postulate (A_2) be satisfied is:

$$\begin{array}{c|c|c|c}
 \oplus & 0 & 1 & \\
 \hline
 0 & 0 & 1 & \\
 1 & 0 & 1 &
 \end{array}, \quad
 \begin{array}{c|c|c|c}
 \oplus & 0 & 1 & \\
 \hline
 0 & 0 & 1 & \\
 1 & 1 & 0 &
 \end{array}, \quad
 \begin{array}{c|c|c|c}
 \oplus & 0 & 1 & \\
 \hline
 0 & 1 & 0 & \\
 1 & 0 & 1 &
 \end{array}, \quad
 \begin{array}{c|c|c|c}
 \oplus & 0 & 1 & \\
 \hline
 0 & 1 & 0 & \\
 1 & 1 & 0 &
 \end{array},$$

i.e., respectively

$$a \oplus b = b, \quad a + b \bmod 2, \quad a + b + 1 \bmod 2, \quad b + 1 \bmod 2.$$

Of these, system (4) is the only one which contradicts both (A_1) and (D) . System (4) is likewise the only possibility for $a \odot b$ in order that both (M_1) and (D) be contradicted. But if (4) be taken for both $a \oplus b$ and $a \odot b$, postulate (D) will be satisfied.

† All the rationals—positive, negative, and zero.